STABILITY OF AN SVIS EPIDEMIC MODEL

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ABSTRACT

The spread of communicable diseases is often described mathematically by compartmental models and applied to control the epidemic. In this paper a nonlinear mathematical deterministic compartmental SVIS model for the dynamics of infectious disease including the role of a preventive vaccine is proposed and analyzed. The model has various kinds of parameter such as natural birth rate, natural death rate and dieses related death rate. Also incoming immigrants are considered in this model. A model for the transmission dynamics of an infectious disease has been presented and analyzed the stability of equilibrium points of this model.

KEYWORDS: Basic Reproduction Number, Diseases Free Equilibrium, Infectious Diseases, Stability Analysis

INTRODUCTION

Mathematical modeling is one of the most important materials to analyze the characteristic of an infectious disease. One of the early triumphs of mathematical epidemiology was the formulation of a simple model by Kermack and McKendrick in 1927 [1]. The Kermack-McKendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population [2]. Various kinds of deterministic models for the spread of infectious disease have been analyzed by mathematical modeling to control the epidemic. Epidemiological models have two kinds of equilibrium points. One of them is disease free equilibrium (DFE) at which the population remains in the absence of disease and other is endemic equilibrium [3]. There are two major types of control strategies available to curtail the spread of infectious diseases: pharmaceutical interventions (drugs, vaccines etc) and non-pharmaceutical interventions (social distancing, quarantine). Vaccination is important for the elimination of infectious disease in pharmaceutical interventions. Arino et al introduced vaccination of susceptible individuals into an SIRS model and also considered vaccinating a fraction of newborns [4]. Buonomo et al studied the traditional SIR model with 100% efficacious vaccine [5]. Effective vaccines have been used successfully to control smallpox, polio and measles. In this paper an SIS type disease has been considered when a vaccination program is in effect. The epidemic models with vaccination have been investigated recently by some authors[8−13]

MODEL FORMULATION

Let $S(t)$ be the number of population who are susceptible to an infection at time t, $I(t)$ be the number of members who are infective at time t, and $V(t)$ be the number of members who are vaccinated at time t. Suppose the total population size at time t is $N(t)$, with $N(t) = S(t) + V(t) + I(t)$. Assume that each infective makes αN contacts sufficient to transmit infection in unit time, where α is a constant. When an infective makes contact, the probability of producing a new infection is S/N, since the new infection can be made only when a contact is made with a susceptible. Thus, the rate of producing new infections is $\alpha N . \dfrac{S}{N} . I = \alpha S I$. Suppose susceptible population is vaccinated at a constant rate ϕ We

assume that there can be disease related deaths as well as natural deaths unrelated to the disease. The population is replenished in two ways, birth and immigration. We assume that all newborns enter the susceptible class at a constant rate Λ and there is a constant incoming flow A of immigrants where some portion of immigrants p, is infective.

In summary, the assumptions we have in this model is as follows:

- $S(t)$, $I(t)$, $V(t)$ and $N(t)$ are the numbers of susceptible, infective, vaccinated, and total population at time t, respectively.

- There is a constant flow A of new members into the population per unit time, where fraction p of immigrants is infective $(0 \le p \le 1)$.

- The vaccine has effect of reducing infection by a factor of σ, so that $\sigma = 0$ means that the vaccine is completely effective in preventing infection, while $\sigma = 1$ means that the vaccine is utterly ineffective.

- ϕ is the rate at which the susceptible population is vaccinated.

- The disease can be fatal to some infective and we define β to be the rate of disease related death.

- There is a constant per capita natural death rate $\mu > 0$ in each class.

- Fraction $\gamma \ge 0$ of infective recovers in unit time.

- αN is the infectious contact rate per person in unit time.

- Λ is the constant natural birth rate, with all newborns coming into the susceptible class.
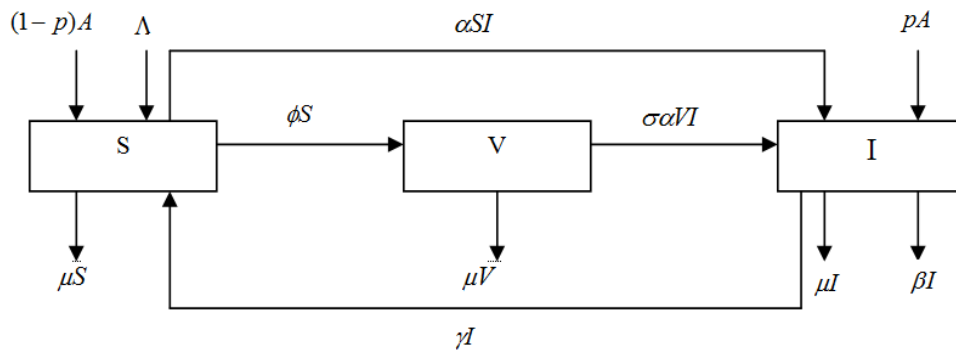
- A is the constant incoming flow of immigrants.

Figure 1: Diagram of SVIS Model

The differential equations of this model are given by

$$
\left.
\begin{aligned}
S' &= (1-p)A + \Lambda - \alpha SI - (\mu + \phi)S + \gamma I \\
I' &= pA + \alpha SI + \sigma \alpha VI - (\mu + \gamma + \beta)I \\
V' &= \phi S - \sigma \alpha VI - \mu V
\end{aligned}
\right\}
\tag{1}
$$

Note that the total population is the sum of three classes: susceptible, infective and vaccinated, i.e.,

$$N(t) = S(t) + V(t) + I(t) \tag{2}$$

So, $N'(t) = S'(t) + V'(t) + I'(t)$

Using (1) we get

$$N'(t) = A + \Lambda - \mu(S + V + I) - \beta I$$

$$\Rightarrow \qquad N' = A + \Lambda - \mu N - \beta I \text{ [using (2)]}$$

We can get an alternate but yet equivalent model by replacing *S* with *N-V-I*. Now the model becomes:

$$\Rightarrow \qquad \left. \begin{array}{l} I' = pA + \alpha I[N - I - (1-\sigma)V] - (\mu + \gamma + \beta)I \\ V' = \phi(N - I) - \sigma\alpha VI - (\mu + \phi)V \\ N' = A + \Lambda - \mu N - \beta I \end{array} \right\} \qquad (3)$$

EQUILIBRIUM CONDITIONS

We can write the equilibrium conditions by letting the right hand side equations of (3) to be zero. The equilibrium conditions are

$$pA + \alpha I[N - I - (1-\sigma)V] - (\mu + \gamma + \beta)I = 0 \qquad (4)$$

$$\phi(N - I) - \sigma\alpha VI - (\mu + \phi)V = 0 \qquad (5)$$

$$A + \Lambda - \mu N - \beta I = 0 \qquad (6)$$

From (6) we get $\quad N = \dfrac{A + \Lambda - \beta I}{\mu}$

Again from (5) we get $\qquad V = \dfrac{\phi(N - I)}{\sigma\alpha I + \mu + \phi}$

$$\Rightarrow \qquad V = \frac{\phi[A + \Lambda - (\beta + \mu)I]}{\mu(\sigma\alpha I + \mu + \phi)} \quad \because N = \frac{A + \Lambda - \beta I}{\mu}$$

Eliminating *N* and *V* by substitution of these expressions into the equation (4), we get the equilibrium condition of the form

$$pA + \alpha I\left[\frac{A + \Lambda - \beta I}{\mu} - I - (1-\sigma)\frac{\phi[A + \Lambda - (\beta + \mu)I]}{\mu(\sigma\alpha I + \mu + \phi)} \right] - (\mu + \gamma + \beta)I = 0$$

Now simplifying by wxMaxima, we obtain an expression involving I of the form

$$EI^3 + BI^2 + CI + D = 0,$$

where, $E = \alpha\sigma(\beta + \mu)$, $\quad B = -(A + \Lambda)\sigma\alpha + (\beta + \mu)(\mu + \sigma\phi) + \sigma\mu(\mu + \gamma + \beta)$

$$C = -\mu p\sigma A - (A + \Lambda)(\mu + \sigma\phi) + \frac{\mu(\mu + \gamma + \beta)(\mu + \phi)}{\alpha}, \ D = -\frac{\mu pA(\mu + \phi)}{\alpha}$$

In this case the model has no dieses free equilibrium.

Theorem (Routh–Hurwitz stability criterion [6]) : Given the characteristics polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + ... + a_n$$

where the coefficients a_i are real constant for $i = 1,2,3,......n$, define the Hurwitz matrices using the coefficients a_i of the characteristics polynomial as follows

$$H_1 = [a_1], \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial equation $P(\lambda) = 0$ are negative or have negative real part iff the determinants of all Hurwitz matrices are positive.

i.e., $\det(H_j) > 0$, for $j = 1,2,3,.......,n$.

When $n = 2$, the Routh–Hurwitz stability criterion simplify to

$$\det(H_1) = a_1 > 0$$

and

$$H_2 = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} = a_1 a_2 > 0$$

or, $a_1 > 0$ and $a_2 > 0$. For polynomial of degree $n = 2,3$ and 4, the Routh–Hurwitz stability criterion is summarized as follows:

$n = 2$: $a_1 > 0$ and $a_2 > 0$.

$n = 3$: $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$.

$n = 4$: $a_1 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$.

EQUILIBRIUMS AND STABILITY ANALYSIS

In order to study the stability of steady states we start a qualitative approach by linearization of (3). Now the jacobean matrix of the system (3) is

$$J = \begin{vmatrix} \dfrac{\partial I'}{\partial I} & \dfrac{\partial I'}{\partial V} & \dfrac{\partial I'}{\partial N} \\ \dfrac{\partial V'}{\partial I} & \dfrac{\partial V'}{\partial V} & \dfrac{\partial V'}{\partial N} \\ \dfrac{\partial N'}{\partial I} & \dfrac{\partial N'}{\partial V} & \dfrac{\partial N'}{\partial N} \end{vmatrix}$$

$$\Rightarrow \quad J = \begin{bmatrix} \alpha[N - (1-\sigma)V - I] - \alpha I - (\mu + \gamma + \beta) & -\alpha(1-\sigma)I & \alpha I \\ -(\phi + \sigma \alpha V) & -\sigma \alpha I - (\mu + \phi) & \phi \\ -\beta & 0 & -\mu \end{bmatrix}$$

Using the equation (4), we can rewrite the jacobean matrix as

$$
J = \begin{bmatrix}
-\dfrac{pA}{I} - \alpha I & -\alpha(1-\sigma)I & \alpha I \\
-(\phi + \sigma\alpha V) & -\sigma\alpha I - (\mu + \phi) & \phi \\
-\beta & 0 & -\mu
\end{bmatrix}
$$

After a complicated computation (with wxMxima), we can obtain its characteristic equation as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 ,$$

where $\quad a_1 = (1+\sigma)\alpha I + \dfrac{pA}{I} + 2\mu + \phi$

$$a_2 = \left(\alpha I + \frac{pA}{I} + \mu\right)(\mu + \varphi + \sigma\alpha I) + \mu\left(\alpha I + \frac{pA}{I}\right) + \alpha\beta I - \alpha I(1-\sigma)(\phi + \sigma\alpha V)$$

$$a_3 = \mu\left(\alpha I + \frac{pA}{I}\right)(\mu + \varphi + \sigma\alpha I) - \mu\alpha I(1-\sigma)(\phi + \sigma\alpha V) + \alpha\beta I(\mu + \sigma\varphi + \sigma\alpha I)$$

By the Routh-Hurwitz Criterion, the steady state is globally stable if and only if

$$a_1 > 0 , \ a_3 > 0 \ \text{and} \ a_1 a_2 > a_3$$

The Figure 2 is the bifurcation curve ϕ vs I (here $\alpha = 0.9$, $\beta = 0.7$, $\gamma = 12$, $\sigma = 0.2$, $\mu = 0.1$, p=0.4, $\Lambda = 3$ and A=2) which demonstrates a case where an equilibrium graph loses its stability as the vaccination rate ϕ, increases and becomes stable again. At the point where it loses local stability first, Hopf-bifurcation occurs and a periodic solution appears for some values of ϕ.
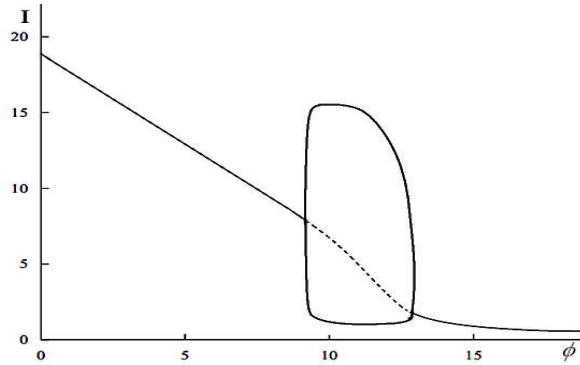

Figure 2: Bifurcation Curve ϕ vs I with Disease-Related Death

THE CASE WHERE THERE ARE NO INFECTIVE IMMIGRANTS

It is worthwhile to consider the case without infective immigrants since in this case the system will have a disease-free steady state that would not exist otherwise. This model was proposed by Kribs-Zaleta and Vekasco-Hernandez [7]. If there is no infective portion from immigrants, i.e. $p = 0$, then our equation becomes

$$S' = A + \Lambda - \alpha SI - (\mu + \phi)S + \gamma I$$
$$I' = \alpha SI + \sigma\alpha VI - (\mu + \gamma + \beta)I$$
$$V' = \phi S - \sigma\alpha VI - \mu V$$

Recall that the total population is the sum of three classes, susceptible, infective and vaccinated, i.e.

$$N(t) = S(t) + V(t) + I(t)$$

Thus it follows that $\qquad N' = S' + V' + I' = A + \Lambda - \mu(S + V + I) - \beta I$

$$\Rightarrow \qquad N' = A + \Lambda - \mu N - \beta I$$

As before we can make a similar transformation by replacing S with N-V-I. Now the model becomes.

$$I' = \alpha(N - V - I)I + \sigma \alpha VI - (\mu + \gamma + \beta)I$$
$$V' = \phi(N - V - I) - \sigma \alpha VI - \mu$$
$$N' = A + \Lambda - \mu N - \beta I$$

$$\Rightarrow \qquad \left.\begin{array}{l} I' = \alpha I\left[N - I - (1 - \sigma)V\right] - (\mu + \gamma + \beta)I \\ V' = \phi(N - I) - \sigma \alpha VI - (\mu + \phi)V \\ N' = A + \Lambda - \mu N - \beta I \end{array}\right\} \qquad (7)$$

We can write the equilibrium conditions by letting the right hand side of (7) to be zero. The equilibrium conditions are

$$\alpha I\left[N - I - (1 - \sigma)V\right] - (\mu + \gamma + \beta)I = 0 \qquad (8)$$

$$\phi(N - I) - \sigma \alpha VI - (\mu + \phi)V = 0 \qquad (9)$$

$$A + \Lambda - \mu N - \beta I = 0 \qquad (10)$$

From (6) we get $\quad N = \dfrac{A + \Lambda - \beta I}{\mu}$

Again from (5) we get $\qquad V = \dfrac{\phi(N - I)}{\sigma \alpha I + \mu + \phi}$

$$\Rightarrow \qquad V = \frac{\phi\left[A + \Lambda - (\beta + \mu)I\right]}{\mu(\sigma \alpha I + \mu + \phi)} \qquad \because N = \frac{A + \Lambda - \beta I}{\mu}$$

Eliminating N and V by substitution of these expressions into the equation (4), we get the equilibrium condition of the form

$$\alpha I\left[\frac{A + \Lambda - \beta I}{\mu} - I - (1 - \sigma)\frac{\phi\left[A + \Lambda - (\beta + \mu)I\right]}{\mu(\sigma \alpha I + \mu + \phi)}\right] - (\mu + \gamma + \beta)I = 0$$

We can further simplify by multiplying $\mu(\sigma \alpha I + \mu + \phi)$ and factoring out a disease free Equilibrium

$$I^* = 0$$

In order to obtain an endemic condition as the quadratic equation for the equilibrium values of I of the form

$$EI^2 + BI + C = 0,$$

where, $\quad E = \alpha \sigma(\beta + \mu), \ \ B = -(A + \Lambda)\sigma \alpha + (\beta + \mu)(\mu + \sigma \phi) + \sigma \mu(\mu + \gamma + \beta)$

$$C = -(A + \Lambda)(\mu + \sigma\phi) + \frac{\mu(\mu + \gamma + \beta)(\mu + \phi)}{\alpha}$$

In order to study the stability of steady states we linearize (4.3.5), obtaining the jacobean matrix.

$$J = \begin{bmatrix} \alpha[N - (1-\sigma)V - I] - \alpha I - (\mu + \gamma + \beta) & -\alpha(1-\sigma)I & \alpha I \\ -(\phi + \sigma\alpha V) & -\sigma\alpha I - (\mu + \phi) & \phi \\ -\beta & 0 & -\mu \end{bmatrix}$$

At the disease free equilibrium $I^* = 0$, The Jacobean becomes

$$J_0 = \begin{bmatrix} \alpha[N - (1-\sigma)V] - (\mu + \gamma + \beta) & 0 & \alpha I \\ -(\phi + \sigma\alpha V) & -(\mu + \phi) & \phi \\ -\beta & 0 & -\mu \end{bmatrix}$$

Now we obtain three real eigenvalues of J_0 as

$$\lambda_1 = -\mu$$

$$\lambda_2 = -(\mu + \phi)$$

$$\lambda_3 = \alpha[N - (1-\sigma)V] - (\mu + \gamma + \beta) = \frac{\alpha(\mu + \sigma\phi)(A + \Lambda)}{\mu(\mu + \phi)} - (\beta + \mu + \gamma)$$

For positive parameters, it is clear that $\lambda_1 < 0$ and $\lambda_2 < 0$. So the disease free equilibrium is asymptomatically stable iff $\lambda_3 < 0$.

$$\Rightarrow \qquad \frac{\alpha(\mu + \sigma\phi)(A + \Lambda)}{\mu(\mu + \phi)} - (\beta + \mu + \gamma) < 0$$

$$\Rightarrow \qquad \frac{\alpha(\mu + \sigma\phi)(A + \Lambda)}{\mu(\mu + \phi)(\beta + \mu + \gamma)} < 1$$

Now we can define the vaccine reproduction number $R(\phi) = \frac{\alpha(\mu + \sigma\phi)(A + \Lambda)}{\mu(\mu + \phi)(\beta + \mu + \gamma)}$. Also by using the endemic equilibrium condition, (8) we can evaluate the Jacobean matrix at endemic equilibriums.

Using the equation (4), we can rewrite the jacobean matrix as

$$J = \begin{bmatrix} -\alpha I & -\alpha(1-\sigma)I & \alpha I \\ -(\phi + \sigma\alpha V) & -\sigma\alpha I - (\mu + \phi) & \phi \\ -\beta & 0 & -\mu \end{bmatrix}$$

with the characteristic equation:

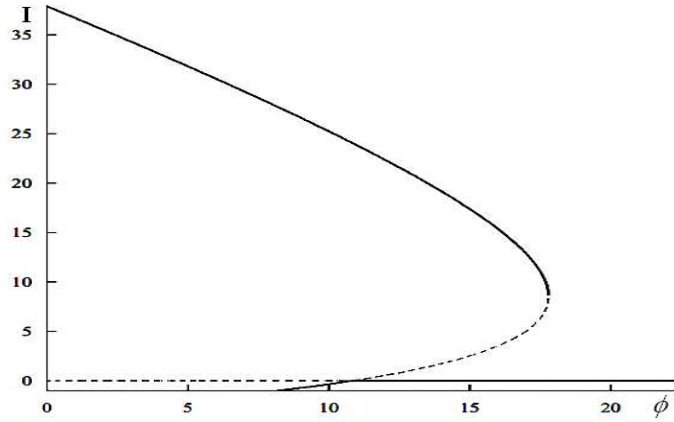$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = (1 + \sigma)\alpha I + 2\mu + \phi$$

$$a_2 = (\alpha I + \mu)(\mu + \varphi + \sigma\alpha I) + \alpha(\mu + \beta)I - \alpha I(1-\sigma)(\phi + \sigma\alpha V)$$

$$a_3 = \mu\alpha I(\mu + \sigma\phi + \sigma\alpha I + \sigma\alpha(\sigma-1)V) + \alpha\beta I(\mu + \sigma\varphi + \sigma\alpha I)$$

By the Routh-Hurwitz Criterion, the endemic steady state is stable iff $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$.

Figure 3: Bifurcation Curve ϕ vs I with No Infective Immigrants

For this model there is a transcritical bifurcation at $\phi = \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$

(This is obtained by solving $R(\phi) = 1$ for ϕ) and this is demonstrated in Figure 3 (here $\alpha = 0.9$, $\beta = 0.15$, $\gamma = 12$, $\sigma = 0.2$, $\mu = 0.1$, p=0, $\Lambda = 3$ and A=2). One can easily see that the lower branch of the bifurcation curve is negative for $\phi < \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$, and coincides with the disease free equilibrium at $\phi = \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$. Also the disease free equilibrium is locally stable for $\phi > \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$ and locally unstable otherwise while the lower endemic equilibrium becomes locally unstable for $\phi > \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$. In summary these equilibriums exchange stability as the endemic equilibrium moves through the diseasefree equilibrium at $\phi = \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$ and there exists only one epidemiologically feasible endemic equilibrium for $\phi < \dfrac{\mu[\mu(\beta+\mu+\gamma) - \alpha(A+\Lambda)]}{\sigma\alpha(A+\Lambda) - \mu(\beta+\mu+\gamma)}$.

CONCLUSIONS

The purpose of this chapter is to take a close look at the endemic behavior of the diseases of SIS type model. To a simple SIS model with vaccination we added the immigration of infective and the disease-related death. As to the contact between infective and susceptible we assume a bilinear incidence. The result of mathematical analysis indicates that a vaccination campaign has an effect of reducing a reproductive number, which means that the average number of secondary infection caused by an average infective becomes smaller when vaccination is in effect. Furthermore, in SVIS model, a vaccination campaign meant to reduce a disease's reproductive number below one, may fail to control the disease when there is a backward bifurcation. Bringing down the vaccination reproductive number just below one may not be good enough to eradicate the disease in such a case. Also if there is no immigration of infective, a typical transcritical bifurcation may be observed. The disease-free equilibrium and endemic one coincide at $R(\phi) = 1$ and they exchange the stability at that point.

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